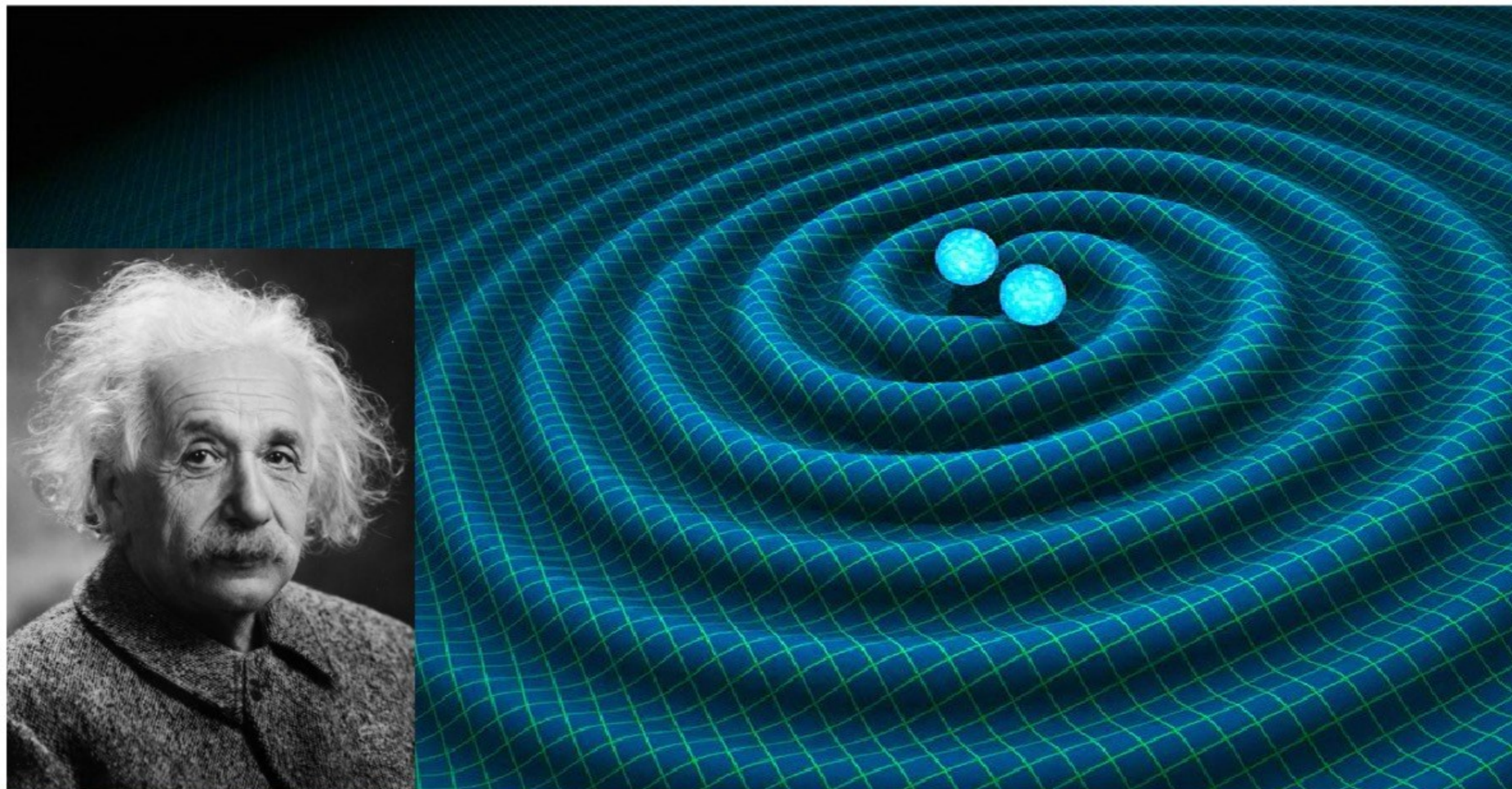


Cosa sono le onde gravitazionali (e i buchi neri...)

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INFN-FLAG



Notazione tensoriale

Coordinate: $x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z)$

Relatività ristretta (1905)

4-Vettori: $V^\mu = (V^0, V^1, V^2, V^3)$

Metrica di Minkowski:

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Lunghezze:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

Covettori:

$$V_\mu = \eta_{\mu\nu} V^\nu = (-V^0, V^1, V^2, V^3)$$

Prodotto scalare:
(lunghezze e angoli)

$$\begin{aligned} V_\mu W^\mu &= V^\mu \eta_{\mu\nu} W^\nu \\ &= -V^0 W^0 + V^1 W^1 + V^2 W^2 + V^3 W^3 \end{aligned}$$

Derivate:

$$\frac{\partial}{\partial x^\mu} = \partial_\mu = (\partial_t, \partial_x, \partial_y, \partial_z)$$

$$\partial^\mu = (-\partial_t, \partial_x, \partial_y, \partial_z)$$

Relatività generale (1915)

Onde Elettromagnetiche

Equazioni di Maxwell (~1860):

$$\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \end{cases}$$



$$\begin{aligned} \vec{E} &= -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla} \times \vec{A} \end{aligned}$$



4-potenziale:

$$A^\mu = (\phi, \vec{A})$$



Tensore di Maxwell:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\begin{cases} \epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho & \mu_0 \epsilon_0 = c^{-2} = 1 \\ \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{cases}$$



4-corrente:

$$J^\mu = (\rho, \vec{J})$$



$$\partial_\alpha F^{\alpha\mu} = -J^\mu$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

Onde Elettromagnetiche

Equazione delle onde EM:

$$J^\mu = 0 \quad \longrightarrow \quad 0 = \partial_\alpha F^{\alpha\mu} = \partial_\alpha \partial^\alpha A^\mu - \partial_\alpha \partial^\mu A^\alpha$$
$$\mu_0 \epsilon_0 = c^{-2} = 1 \quad \quad \quad = -(\partial_t^2 - c^2 \nabla^2) A^\mu - \partial^\mu \partial_\alpha A^\alpha$$

Relatività ristretta

Invarianza di gauge:

$$A_\mu \rightarrow \bar{A}_\mu = A_\mu + \partial_\mu \Lambda \quad \longrightarrow \quad F_{\mu\nu} \rightarrow \bar{F}_{\mu\nu} = F_{\mu\nu}$$

1) gauge di Lorentz:

$$\partial_\mu \bar{A}^\mu = 0 \quad \longleftarrow \quad \partial_\alpha \partial^\alpha \Lambda = -\partial_\alpha A^\alpha$$

2) completamento del gauge:

$$\bar{A}_\mu \rightarrow \tilde{A}^\mu = \bar{A}_\mu + \partial_\mu \bar{\Lambda} \quad \text{con} \quad \partial^\mu \partial_\mu \bar{\Lambda} = 0$$

4 g.d.l. - 2 condizioni di gauge = 2 g.d.l.

A^μ

Λ

Onde Elettromagnetiche

Onde EM piane:

Vettore d'onda $k^\mu = (k, 0, 0, k)$

$$k^\mu k_\mu = 0$$

$$A^\mu = \epsilon^\mu e^{i k_\alpha x^\alpha} = \epsilon^\mu e^{i k z - i k t}$$

1) gauge di Lorentz: $0 = \partial_\mu A^\mu = (k \epsilon^z - k \epsilon^t) e^{i k z - i k t}$

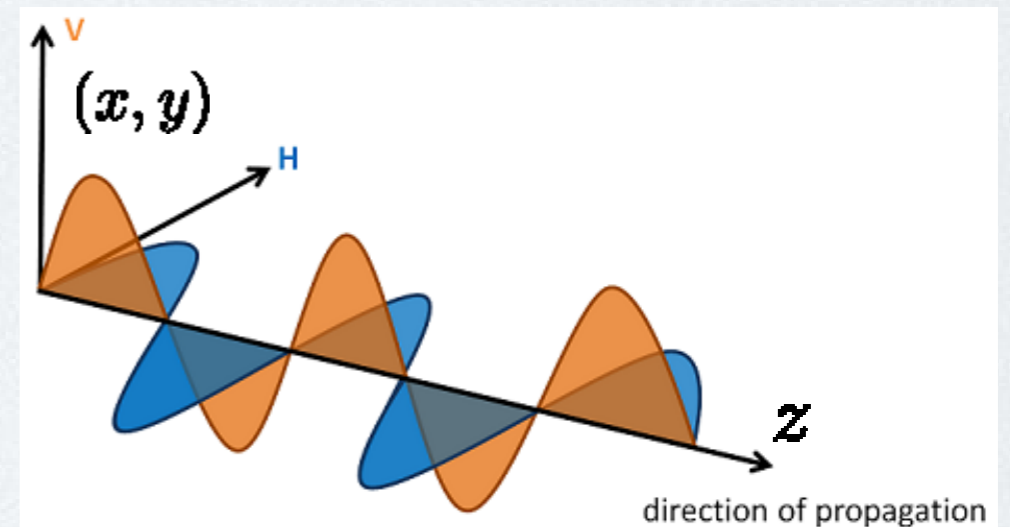
$$\epsilon^z = \epsilon^t \quad \longrightarrow \quad k_\mu \epsilon^\mu = -k \epsilon^t + k \epsilon^z = 0$$

2) completamento del gauge: $\partial_\mu \partial^\mu \Lambda = 0$

$$\Lambda = i \frac{\epsilon^z}{k} e^{i k z - i k t}$$

$$A_\mu + \partial_\mu \Lambda \sim (-\epsilon^z, \epsilon^x, \epsilon^y, \epsilon^z) + (\epsilon^z, 0, 0, -\epsilon^z) \\ = (0, \epsilon^x, \epsilon^y, 0)$$

$$\vec{E} \simeq k (\epsilon^x, \epsilon^y, 0) e^{i k z - i k t}$$

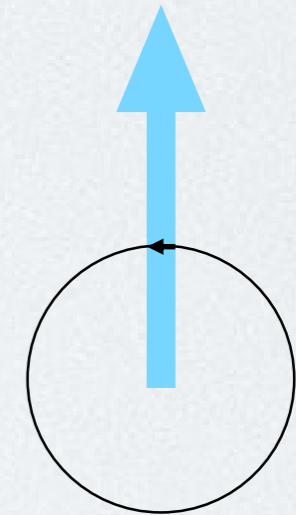
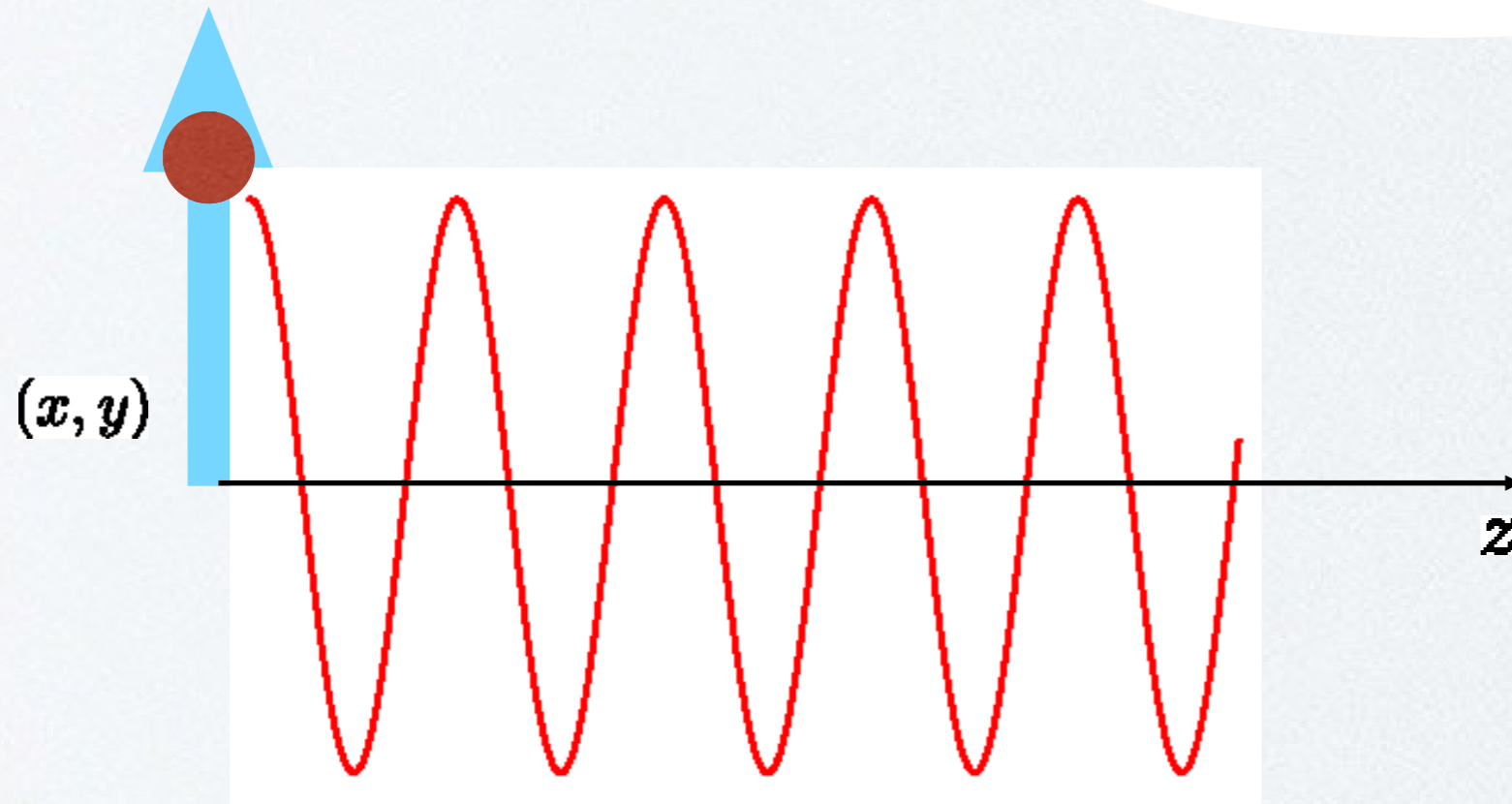


Onde Elettromagnetiche

Onde EM e dipolo:

$$\partial_\alpha F^{\alpha\beta} = \partial_\alpha \partial^\alpha A^\beta = -J^\beta \sim \rho n^i$$

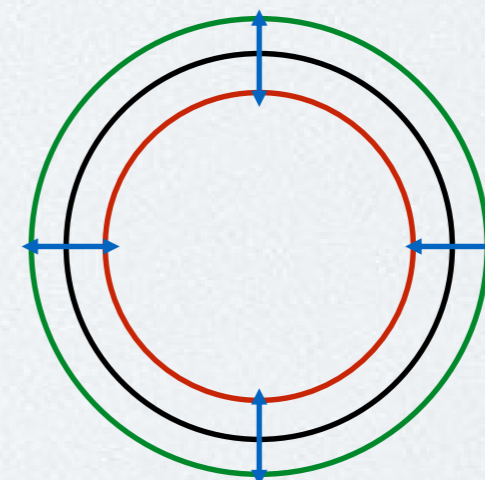
$\beta = x, y$



$$\text{spin} = 1/(1 \text{ giro}) = 1$$

Antenna ~ Sorgente ~ Rivelatore

Non esistono onde EM di monopolo:



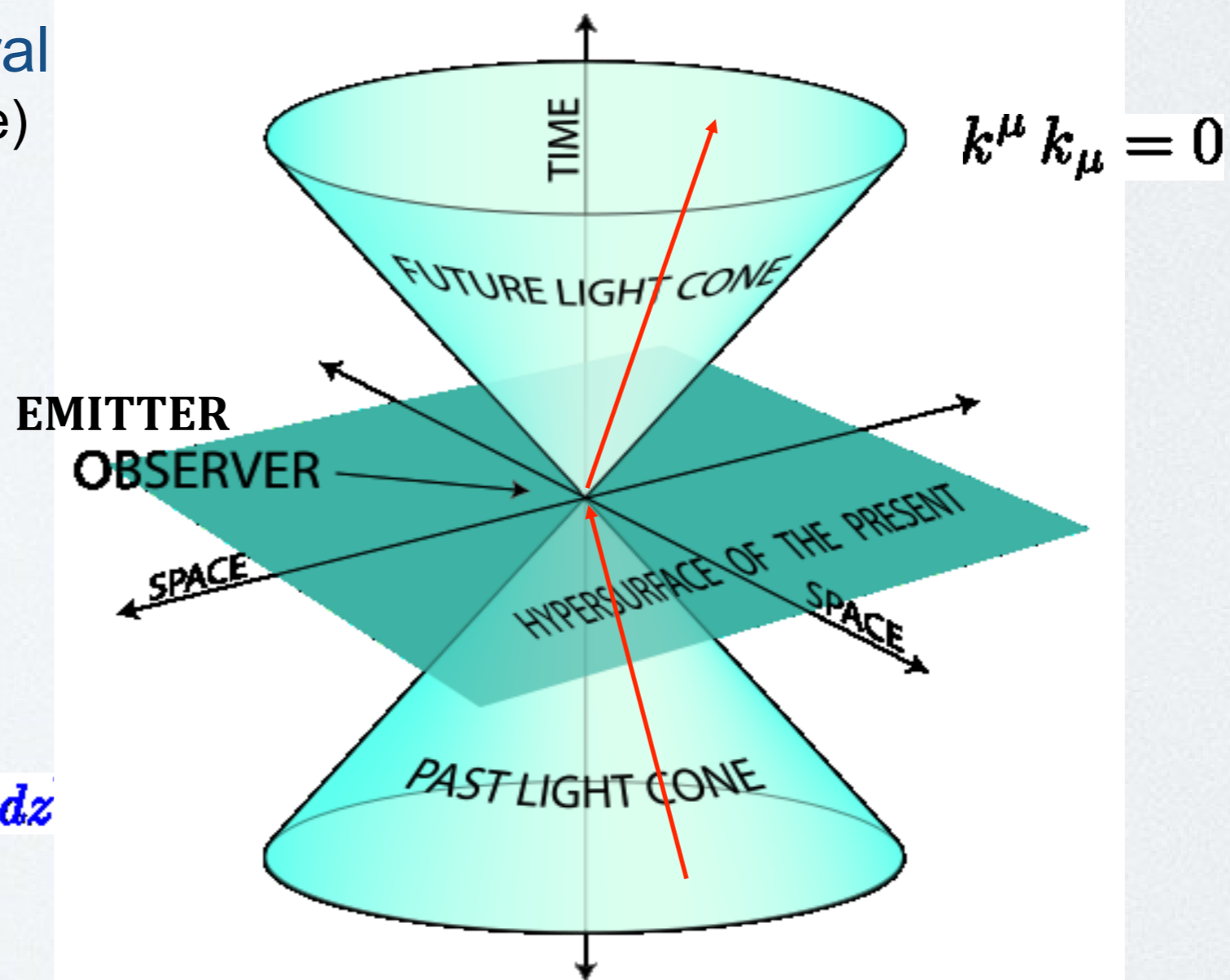
$$\text{spin} = 0$$

Coni Luce in Relatività Ristretta

Distanze spazio-temporal
(struttura causale)

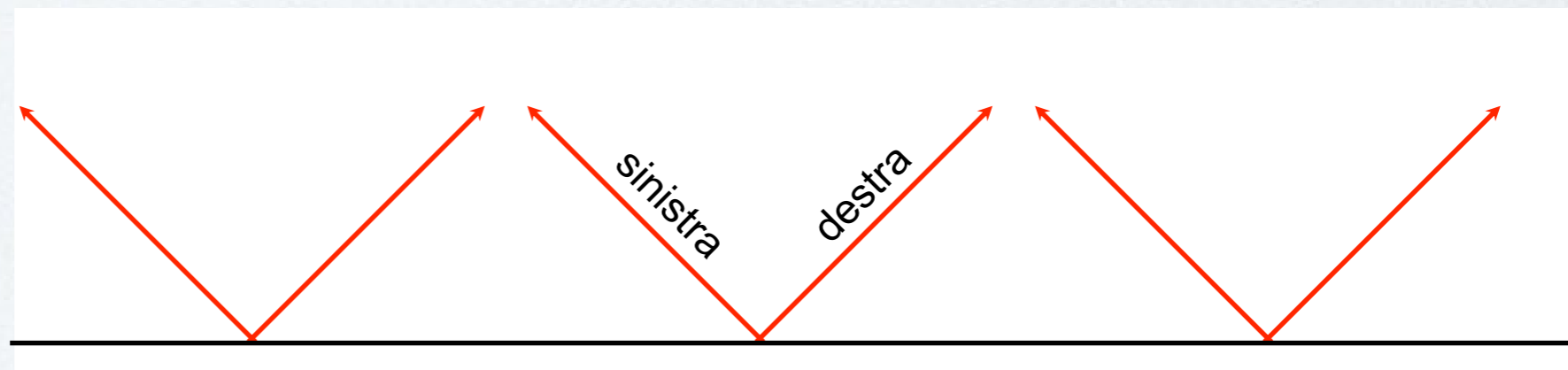
$$x^\mu = (t, x, y, z)$$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$



Spazio-tempo piatto

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



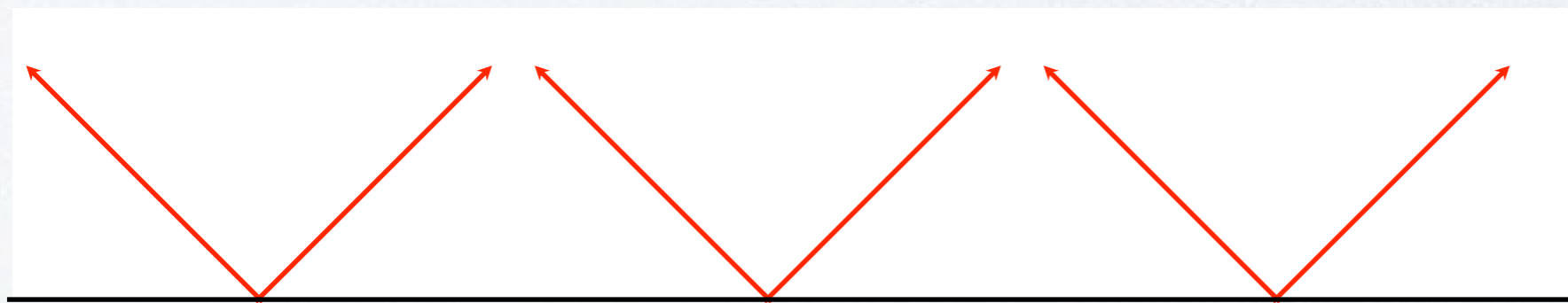
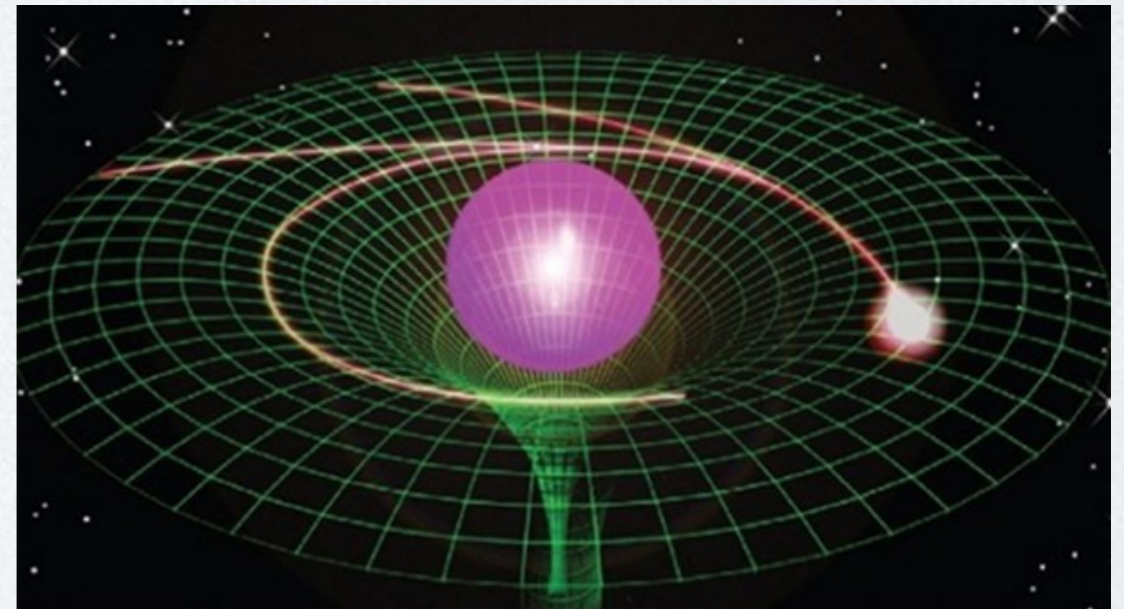
Coni Luce in Relatività Generale

Equazioni di Einstein (1915):

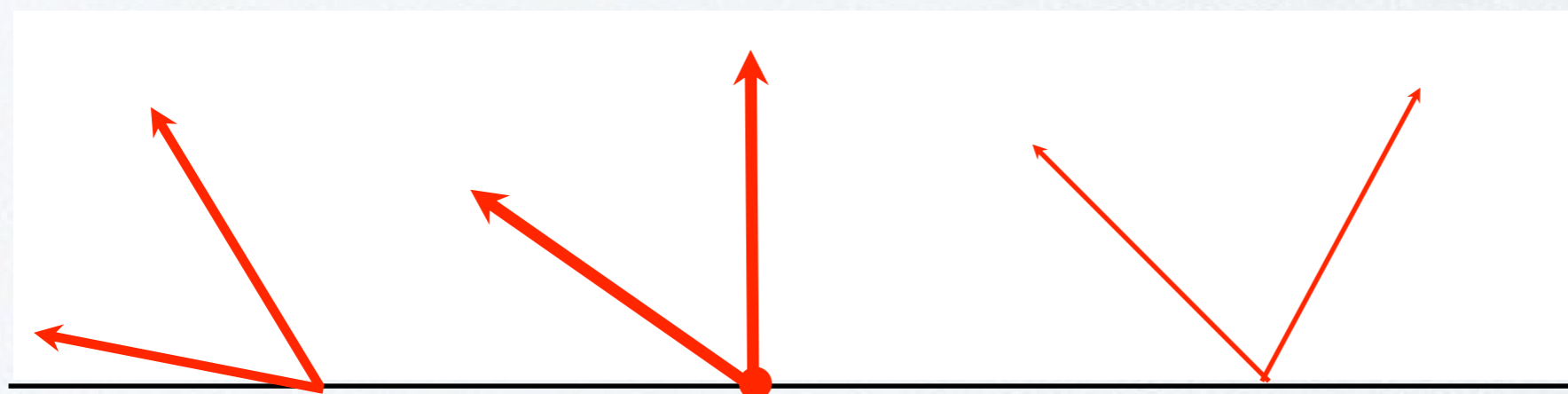
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

↑
Curvatura spazio-tempo

↑
Energia-Impulso



spazio-tempo piatto



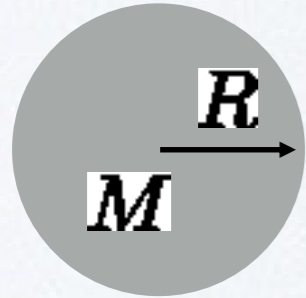
Orizzonte

spazio-tempo curvo
(buco nero)



Buchi neri

Orizzonte (“Newtoniano”) ~ velocità di fuga = ($c = 1$) (Michell/Laplace: XVIII secolo)

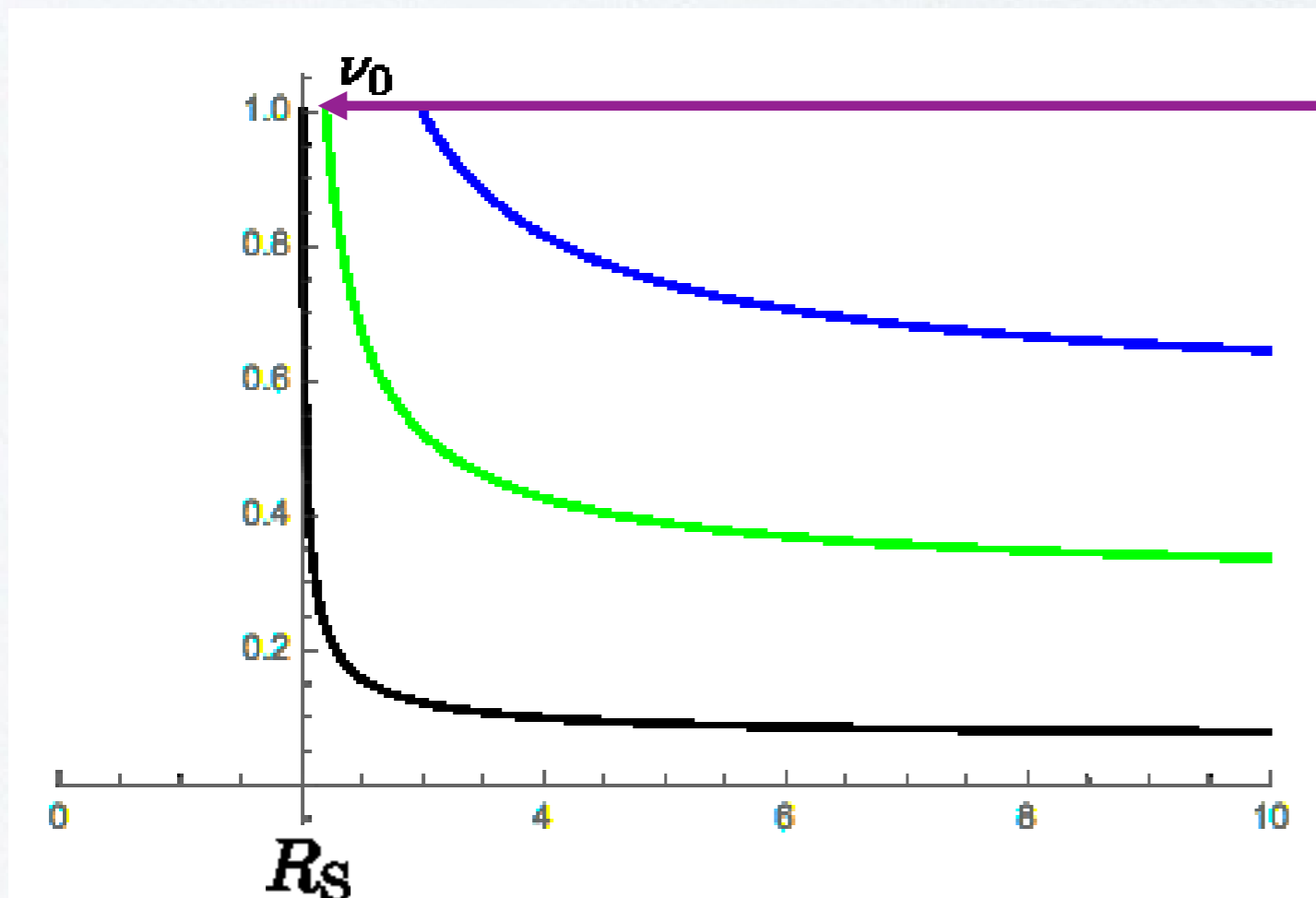


$$\frac{1}{2} m (v_f^2 = 1) - G_N \frac{M m}{R} = 0 \Rightarrow R = R_S = 2 G_N M$$

Orizzonte ~ redshift illimitato + dilatazione tempi illimitata

(Buco Nero ~ 1968)

(Stella Congelata < 1968)



Sonda

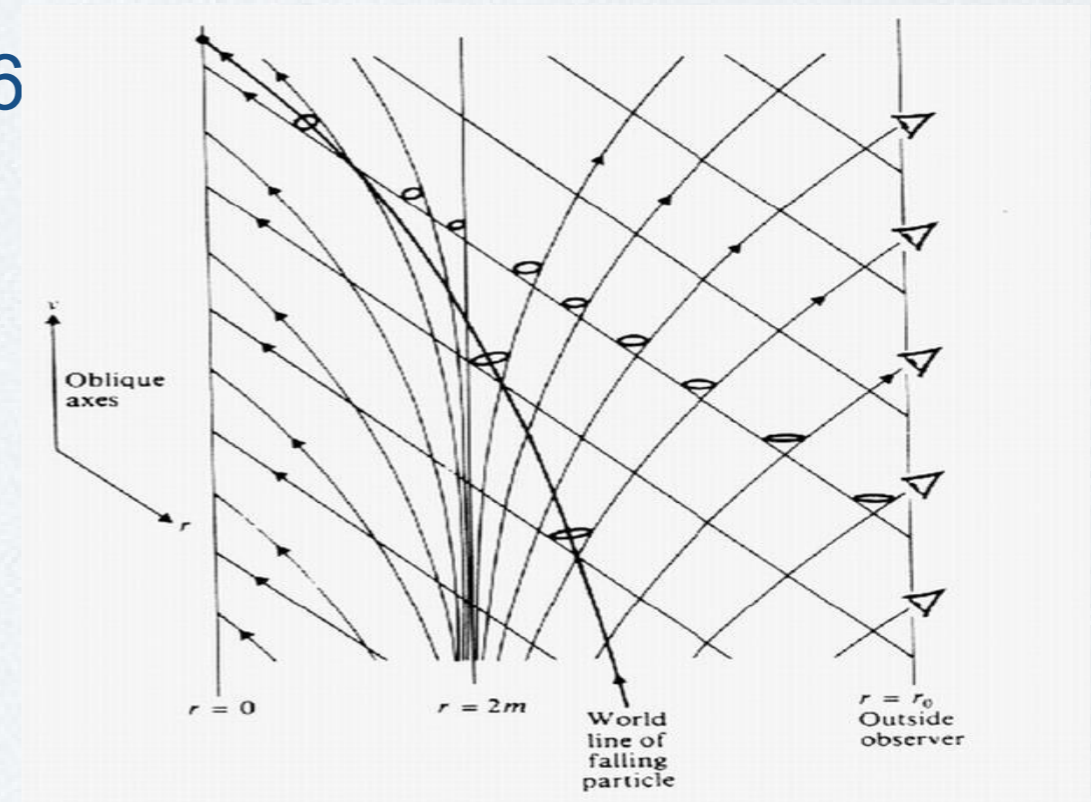
$$\frac{\nu}{\nu_0} = \sqrt{\frac{1 - R_S/r_0}{1 - R_S/r}}$$

Buchi neri

Schwarzschild: buco nero sferico (1916)

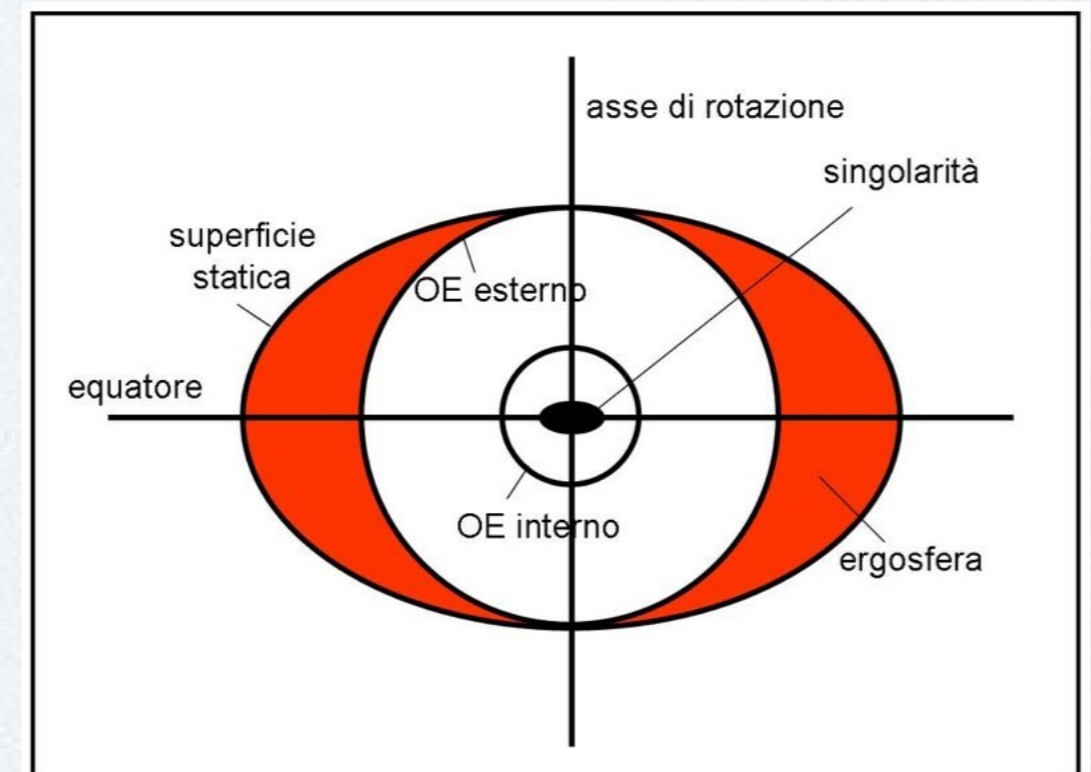
Orizzonte degli eventi = redshift "infinito"

$$R_S = 2 G_N M$$



Kerr: buco nero rotante (1963)

Orizzonte degli eventi + Ergosfera



Buchi neri

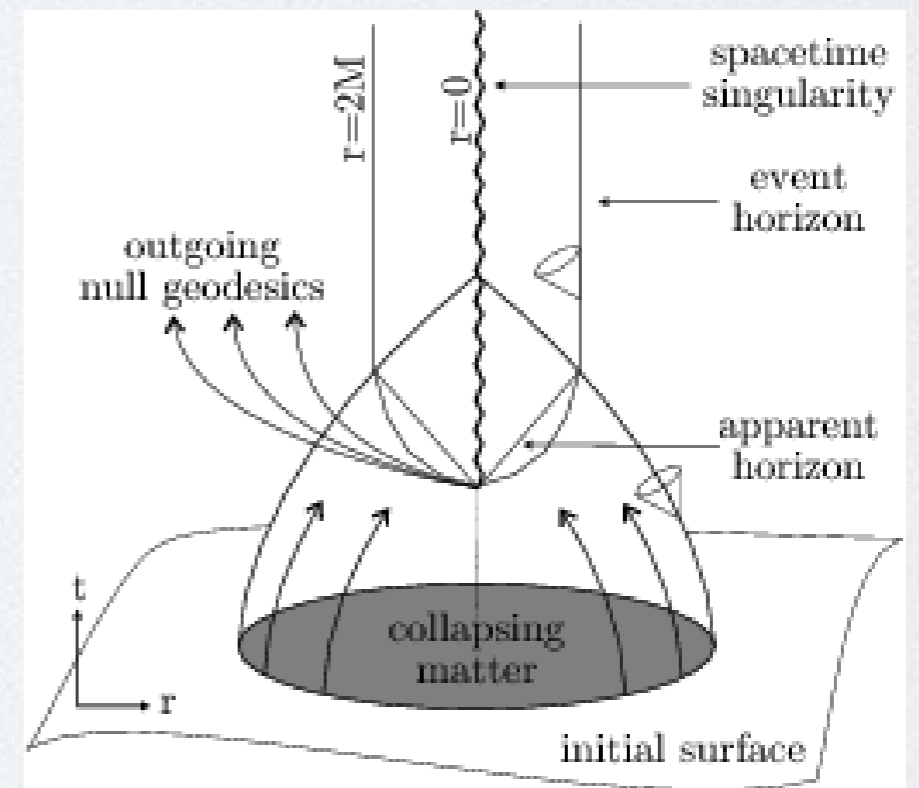
Collasso gravitazionale:

Supernovae

Fusione di stelle (di neutroni)

...

Fluttuazioni primordiali



Attrazione gravitazionale supera forze repulsive (Oppenheimer e c. ~ 1940)

Equazione di Tolman-Oppenheimer-Volkoff e limite di compattezza (M/R)

Onde Gravitazionali

Equazioni di Einstein

1) $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu}$

2) $\sim \partial_\alpha \partial_\beta g_{\mu\nu}$

Lunghezze!

3) Forze gravitazionali "apparenti" $\sim \partial_\alpha g_{\mu\nu}$

5) Equazioni non-lineari
(onde...?)

vs

Equazioni di Maxwell

$$\partial_\alpha F^{\alpha\mu} = -J^\mu$$

Sorgenti

$$\sim \partial_\alpha \partial_\beta A^\mu$$

No fisica (classica)

Forze EM $\sim \partial_\alpha A^\mu$

Equazioni lineari
(principio di sovrapposizione)

Potenziali

Onde Gravitazionali

Equazioni di Einstein linearizzate (1916-...): $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$ $\varepsilon |h_{\mu\nu}| \ll 1$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \varepsilon [\partial_\alpha \partial^\alpha h_{\mu\nu} + (\text{gauge})] + O(\varepsilon^2)$$

“Invarianza di gauge” (cambio di coordinate!):

$$x^\mu \rightarrow x^\mu + \xi^\mu(x)$$

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$



scelta di coordinate!

$$(\text{gauge}) = 0$$

$$\partial_\alpha \partial^\alpha h_{\mu\nu}^{\text{TT}} = (\partial_t - \nabla^2) h_{\mu\nu}^{\text{TT}} \simeq 0$$

10 g.d.l. - 8 condizioni di gauge = 2 g.d.l.

$h_{\mu\nu}$

ξ^μ

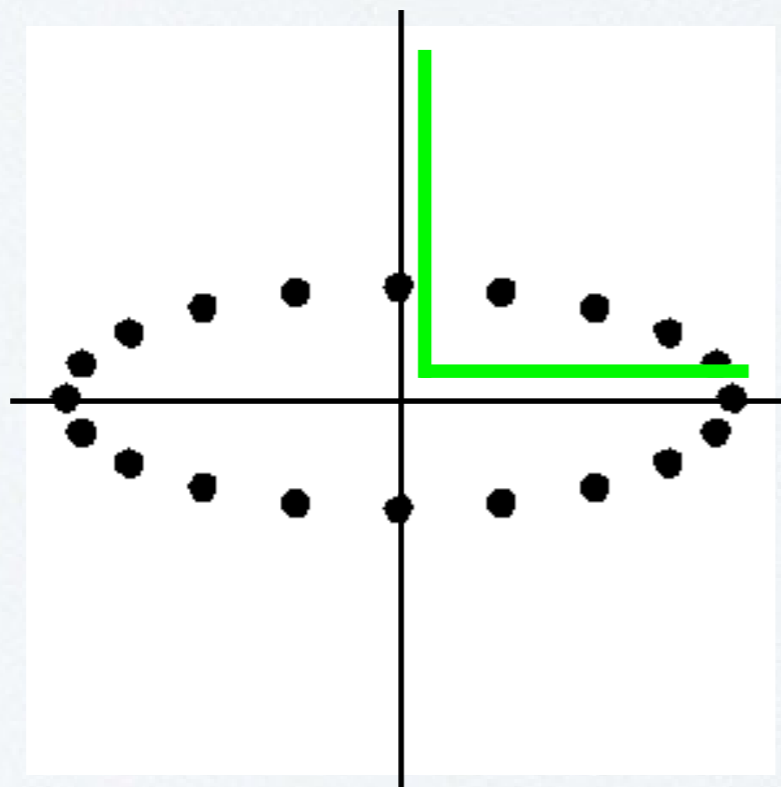
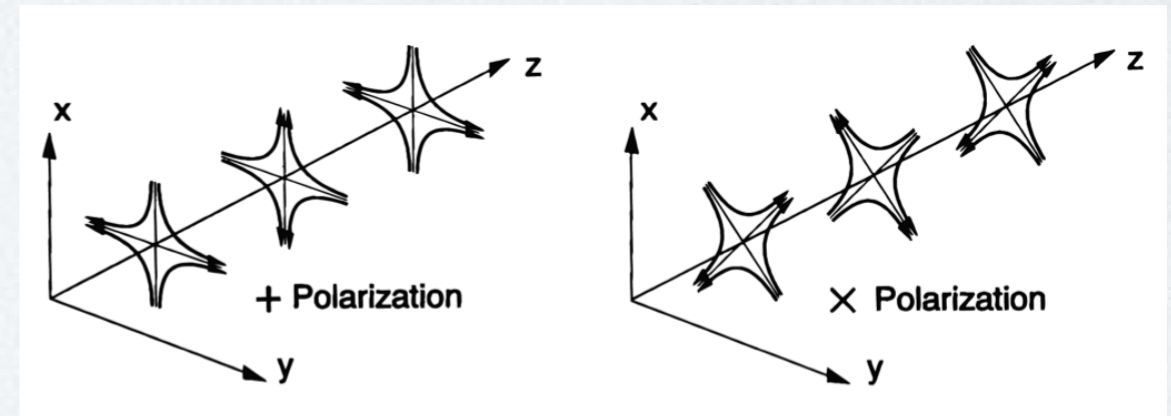
Onde Gravitazionali

Gauge trasversale a traccia nulla (TT):

$$k^\mu = (k, 0, 0, k) \quad k^\mu k_\mu = 0$$

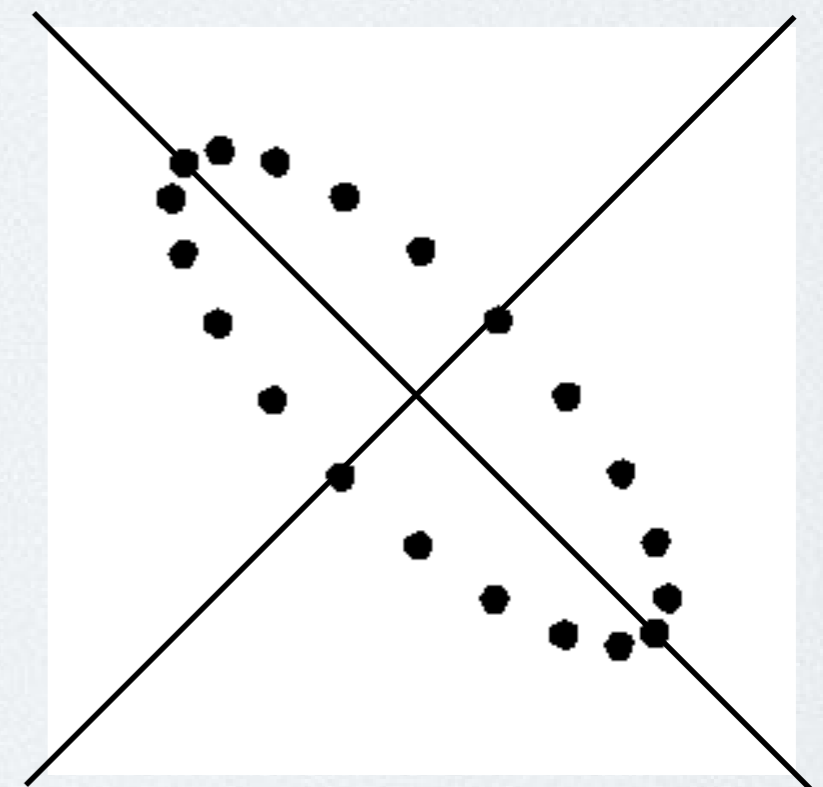
$$h_{\mu\nu} = h_{\mu\nu}^{\text{TT}} e^{i k_\alpha x^\alpha}$$

$$h_{\mu\nu}^{\text{TT}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



Rivelatore

- masse libere
- righelli "fissi"



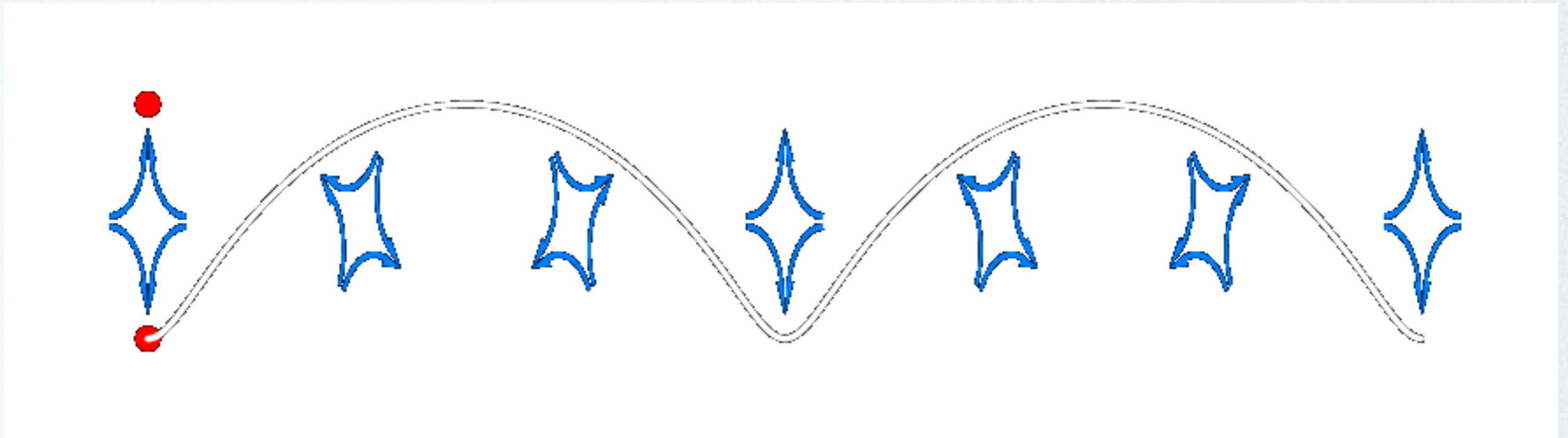
Onde Gravitazionali

Onde gravitazionali e quadrupolo:

$$\partial_\alpha \partial^\alpha h_{\mu\nu}^{\text{TT}} \simeq 8\pi G_{\text{N}} T_{\mu\nu} \sim \rho n^i n^j$$

$i, j = x, y$

Sistema binario (2 masse “puntiformi” uguali):

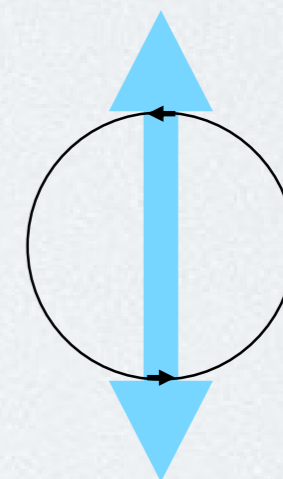
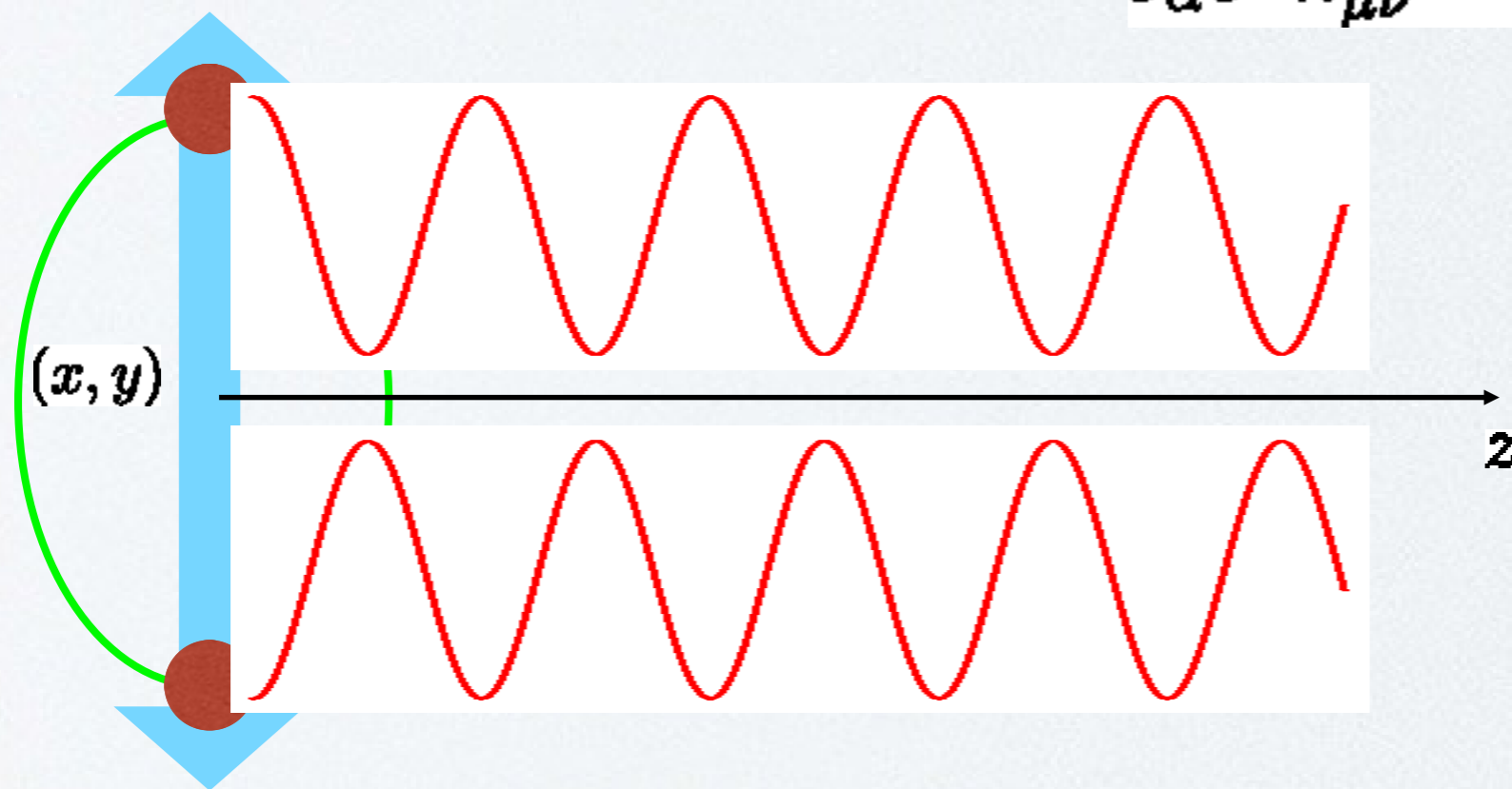


N.B. Monopolo e dipolo = 0

Onde Gravitazionali

Onde gravitazionali e quadrupolo:

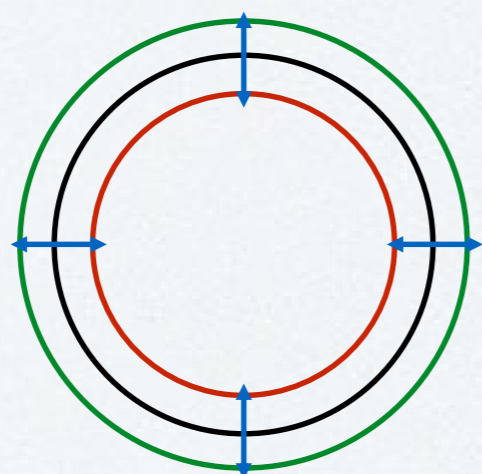
$$\partial_\alpha \partial^\alpha h_{\mu\nu}^{\text{TT}} \simeq 8\pi G_N T_{\mu\nu} \quad \beta = x, y$$



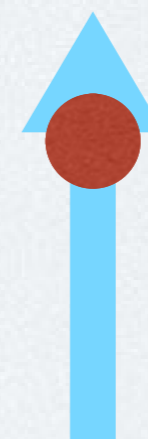
$$\text{spin} = 1 / (1/2 \text{ giro}) = 2$$

Sorgente

Non esistono onde gravitazionali di monopolo e dipolo!

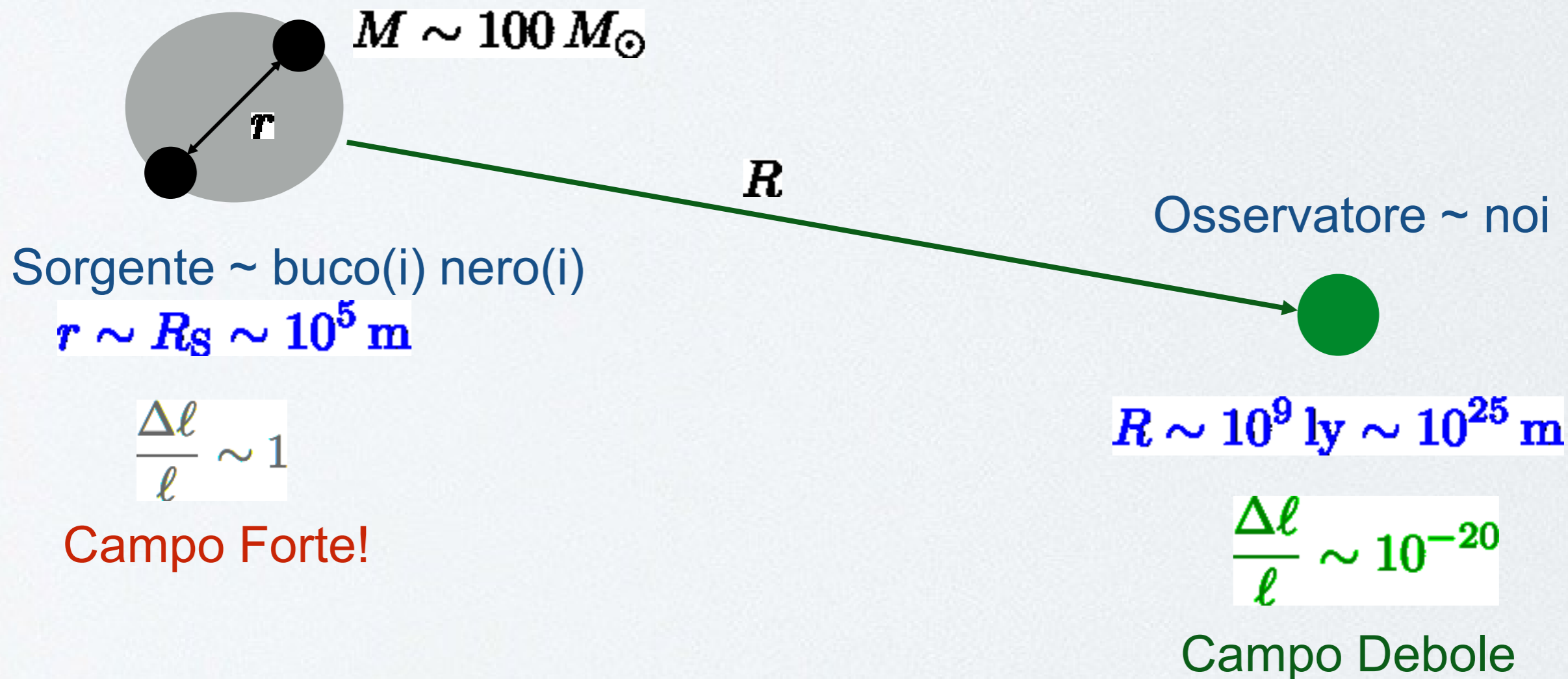


spin = 0



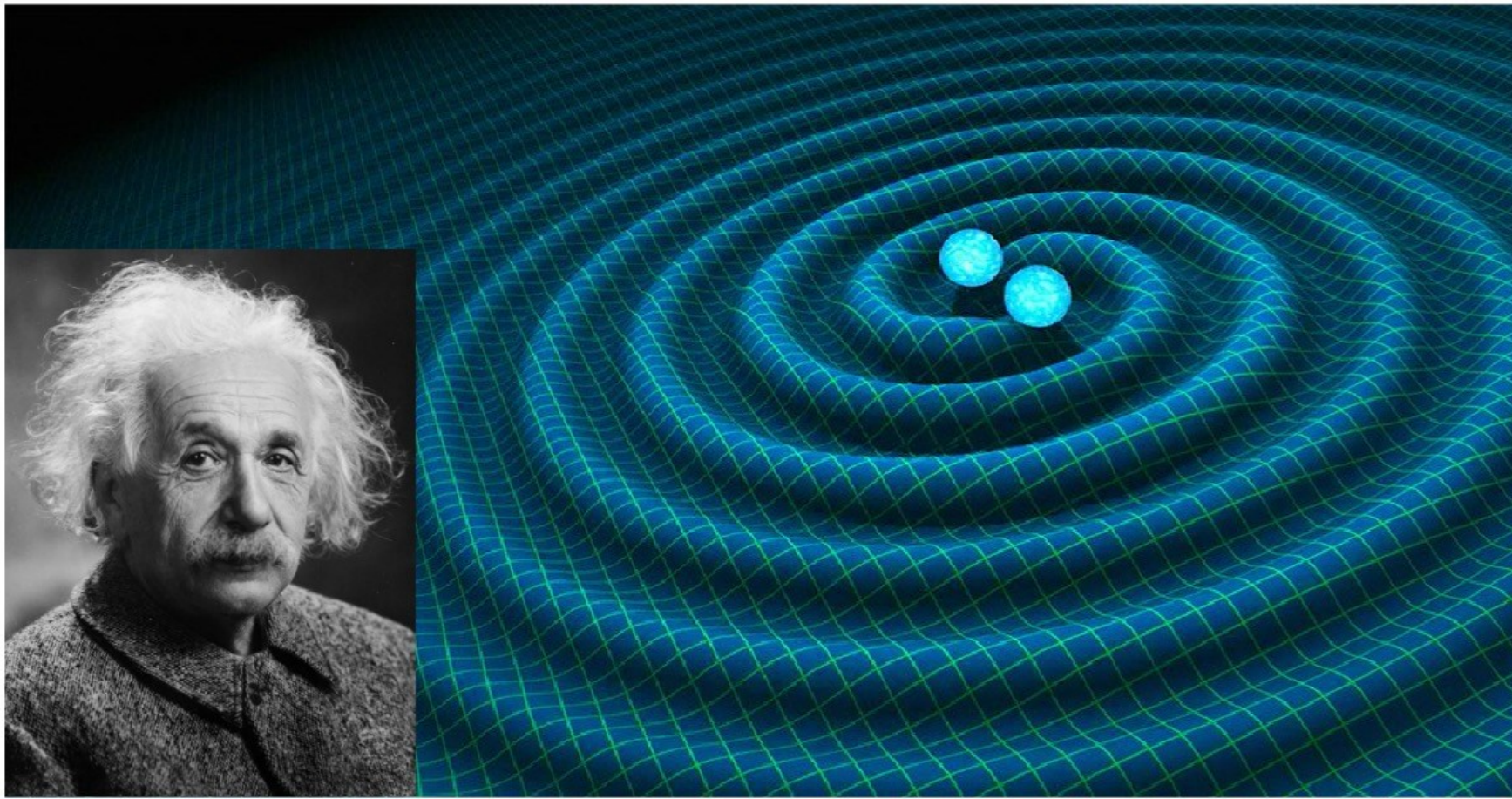
spin = 1

Onde Gravitazionali



Deformazione (strain):

$$\frac{\Delta l}{l} \simeq h_+ \simeq \frac{G_N^2 M^2}{r R} \simeq \frac{R_S}{r} \frac{R_S}{R}$$



Einstein 1915 (1916-37) - LIGO 2015
(2016)

